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Conjugate-flow theory for heterogeneous compressible fluids, with application to non-uniform suspensions of gas bubbles in liquids

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Conjugate flows have been defined generally as flows uniform in the direction of streaming that separately satisfy the relevant hydrodynamical equations, so allowing a transition from one flow to its conjugate to be consistent with mass and energy conservation. In previous studies of various examples, certain general principles have been found to apply to conjugate flows: in particular, one in a pair of such flows is subcritical (subsonic) and the other supercritical (supersonic), the former having greater flow force (i.e. momentum flux plus pressure force). In this paper these principles are confirmed in another field of application, for which the theory of conjugate flows takes a novel course.

The theoretical model defined in § 2 consists of a straight duct of arbitrary cross-section filled with a perfect fluid whose constitutive properties vary with cross-sectional position, and whose primary, prescribed flow is axial with a velocity distribution that may be non-uniform. In § 3 the possibility of a conjugate flow in the same duct is investigated, and its principal properties relative to those of the primary flow are deduced from certain simple inequalities between integrals over the cross-section. A Lagrangian description of the conjugate flow is essential, but the properties in question are established without the necessity of determining this flow explicitly. At the end of § 3, a modification of the model is discussed accounting for dissipative, flow-force conserving transitions (shocks). The application of the theory to flows of non-uniform suspensions of gas bubbles is considered in § 4.

1. Introduction

It was suggested in a recent paper (Benjamin 1971, 2nd footnote to p. 590) that general conjugate-flow principles should apply to flows of liquid–gas mixtures, even when the properties of the composite fluid are not uniform over the flow cross-section.† For *homogeneous* suspensions of small gas bubbles in liquids

† As an example of how such non-uniformities can arise naturally, consider the situation where a profusion of small gas bubbles is suspended in a liquid flowing along a horizontal tube, and originally the suspension is uniform. Due to migration of the bubbles under gravity, the local average concentration of gas in the suspension will gradually depart from uniformity with increasing distance along the tube, becoming smallest in the lowest region of the cross-section, and the local distribution of bubble sizes will also become dependent on cross-sectional position.

a theory of normal shock waves is known, having been initiated by Ackeret (1930) and developed by Campbell & Pitcher (1958), and a recent study by van Wijngaarden (1968) has pointed out a detailed correspondence between this problem and the problem of open-channel flows that provides the standard exemplification of conjugate-flow principles. If the assumption of homogeneity is relaxed, however, the formal analogy with the water-wave problem disappears and the relevance of these principles ceases to be obvious. The expectation that they should still apply can be supported, as in the cited paper, by certain abstract considerations about nonlinear flow problems, but a specific treatment of the new problem is clearly more satisfactory from a practical standpoint. In the present paper the usual properties of conjugate flows are confirmed in respect of non-uniform suspensions of gas bubbles, being shown to emerge in a novel way from the equations of steady motion.

The required theory for gas-bubble suspensions is presented as an instance of a more general theory, which conceivably may have other practical applications. In § 2 the generalized version of the problem is introduced, allowing for a duct of arbitrary cross-section filled with a fluid whose constitutive properties vary arbitrarily over the cross-section. A primary flow of this fluid is defined, having an axial velocity that may be non-uniform, and a criterion is explained whereby the flow can be classified as subsonic or supersonic. The main theoretical results are derived in § 3, establishing properties of a flow that is conjugate to the primary one in the sense that it may occupy the same duct and satisfies conditions of mass and energy conservation in relation to the primary flow. A note on the practical interpretation of the established conjugate-flow properties is included near the end of § 3, and in conclusion an adaption of the theory is proposed which serves as a rough model for shock-wave transitions that conserve flow force but are dissipative. Then, in § 4, the details of the application to non-uniform suspensions of gas bubbles are presented.

The material of this paper is essentially a contribution to nonlinear long-wave theory, and in the main no attention is paid to interesting frequency-dispersive effects that appear to be typical of the flow systems in question. To conclude the discussion in § 5, however, a characteristic-value problem is noted relating to the dispersion of infinitesimal waves unrestricted in length. The dispersive effects thus represented, and particularly their interactions with mild nonlinear effects, seem to be a deserving subject for further study.

To understand the intention of the following analysis, one must appreciate the complexity of the physical situations towards whose clarification this work is directed. In practice flows of bubble suspensions are likely to depend significantly on several factors not treated here, notably dissipation due to relative motion between the bubbles and the liquid, and a comprehensive theory seems out of the question. But simplified models such as the present one can still offer insight into complicated phenomena even when there is little hope of analysing them accurately. As exemplified by its application to the vortex breakdown phenomenon (Benjamin 1962, 1965, 1967), conjugate-flow theory provides a rationale for a class of nonlinear effects hardly amenable to precise analysis, although its practical value generally depends on broad physical reasoning with which it can be coupled. Some reassurance that in the present application the theory is well founded, despite its drastic simplifications, is given by the argument at the end of § 3 where a hypothetical allowance for dissipation is shown to have plausible consequences.

2. Specifications of the generalized problem

We take Cartesian co-ordinates (x, y, z) with z along the duct, whose cross-section is denoted by σ. The cross-sectional area is denoted by $\mathscr{A}$, thus

$$\mathscr{A} = \int_\sigma dx\, dy.$$

The fluid filling the duct is supposed to be inviscid, and the effect of gravity on it is ignored. In the primary state of the fluid the pressure p is therefore a constant, which we take to be zero without loss of generality, and the other fluid properties are independent of z. It is assumed that the density ρ of a sample of fluid taken from any point of the cross-section depends only on pressure, but the relationship between ρ and p varies with position in σ. That is, we are given the function

$$\rho = R(p, x, y), \quad (x, y) \in \sigma, \tag{2.1}$$

such that $R(0, x, y)$ is the density in the primary state and, if the fluid is disturbed from this state, its density is specified by $R(p, x, y)$ with x, y as Lagrangian co-ordinates referring to particular fluid particles. Correspondingly, we have as a given function

$$\int_0^p \frac{dp'}{\rho(p')} = H(p, x, y), \quad (x, y) \in \sigma. \tag{2.2}$$

Hereafter these functions will generally be written simply as $R(p)$ and $H(p)$, but will always be implied to depend on (x, y).

The preceding assumption, that the density of a fluid sample depends only on pressure, supersedes the thermodynamic considerations needed to describe the general behaviour of a compressible fluid, and the meaning of this simplification deserves to be stressed. We recall that for steady flows the condition of energy conservation may be written

$$\delta(\tfrac{1}{2}u^2 + E + p/\rho) = 0, \tag{2.3}$$

where δ denotes variation along a streamline, $\frac{1}{2}u^2$ the kinetic energy and E the internal energy of the fluid per unit mass. The sum $E + (p/\rho)$ is termed the enthalpy. Now, it is implied by (2.1) that changes of state in the fluid follow a prescribed *reversible* process; and if we further assume the process to be adiabatic,†

† As regards the application to suspensions of gas bubbles, the implication of this assumption is not necessarily that the gas in each bubble is compressed or expanded adiabatically: it is the composite fluid that is understood to retain heat. In fact, for suspensions of fairly small bubbles undergoing flow processes that are not exceptionally rapid, it is a good approximation to assume that thermal equilibrium is maintained between the components (see § 4). Since the heat capacity of the liquid enormously exceeds that of the suspended gas, compression of the gas in each bubble is virtually isothermal and the heat generated by it is absorbed in a surrounding layer of liquid. The changes of state in the composite fluid are then effectively reversible and adiabatic in the present sense.

then changes of internal energy are accountable only to the mechanical work of compression: thus
$$dE = -pd(1/\rho).$$
It follows that the function H defined by (2.2) is identifiable with enthalpy relative to the primary state, and the energy equation becomes
$$\delta(\tfrac{1}{2}u^2 + H) = 0, \tag{2.4}$$
which is a relation between mechanical properties alone.

Equation (2.4) is to be the basis for precise deductions about idealized conjugate flows, but in respect of practical applications the possibility of its being in error to some extent has to be acknowledged. As already mentioned, internal friction is particularly relevant as a possible source of error in flows of gas-bubble suspensions, because relative motion between the bubbles and the liquid is induced by any pressure gradient to which a suspension is subjected. To allow for such irreversible effects, a quantity Δ may be defined as the surplus of internal energy above the level acquired through a reversible adiabatic process as implied by (2.1). Accordingly Δ is added within the brackets in equation (2.4), being describable as the waste of available energy. [In thermodynamics Δ is usually expressed as $\int T ds$, where T is temperature and s entropy; but this notion does not seem additionally helpful in the present context. For instance, if a flowing suspension of gas bubbles maintains thermal equilibrium and the predominant cause of energy waste is internal friction, then the temperature of the liquid is hardly relevant to the essentially mechanical problem: the thermodynamic interpretation, that the frictional losses heat up the liquid by a minute amount and entropy is increased, is obviously a side-light.]

The most likely practical application of the theory is to the case of a plane shock wave advancing with constant speed C into fluid at rest. By taking a frame of reference moving with the wave, we may model this situation as a steady flow in which the fluid approaches with uniform velocity C. But the present treatment will extend to the case where the undisturbed fluid has a non-uniform axial velocity $U(x, y)$. In a frame moving with a wave that propagates against the flow, the primary velocity is then
$$W(x, y) = C + U(x, y).$$
We shall hereafter treat $W(x, y)$ as a given property of our steady-flow model, with further reference to this interpretation, and we assume that $W > 0$ throughout σ. The results obtained on this basis apply directly, of course, to the practical case of stationary shocks or other transitions on a given stream.

Definition of subsonic and supersonic

The theory to be developed in § 3 relies on an idea proposed by Benjamin (1962, 1965) for generalizing the classification of subcritical and supercritical states of steady flow – called here subsonic and supersonic since sound waves are indeed in question. This idea becomes expedient when the primary axial velocity is non-uniform, so that there is no frame of reference in which the fluid appears stationary.

Let $W(x,y)$ be the velocity in the $+z$ direction, and suppose that a free wave of extreme length is superposed without energy loss on the flow. The wave may propagate with either of two velocities, say c_+ and $c_- < c_+$, reckoned positively in the z direction. Propagating with the flow its velocity will be $c_+ > 0$, which, because of the flow's convective action, will be greater than the velocity of a long wave relative to the same fluid at rest. Propagating against the flow its velocity will be c_-, which may be either positive or negative, the latter if the convective action is not too great and consequently the wave makes headway upstream. The generalized Mach number may be defined by

$$N = (c_+ + c_-)/(c_+ - c_-), \tag{2.5}$$

which in the case $W = \text{const.}$ reduces to the familiar definition

$$N = W/c$$

since $c_+, c_- = W \pm c$, where c is the long-wave speed relative to the fluid at rest.

The given flow $W(x,y)$ is supersonic if $c_- > 0$ and so $N > 1$. This condition may also be understood to mean that a flow with the *reduced* velocity $W(x,y) - c_-$ is exactly sonic, for this is the steady flow observed in a frame of reference moving downstream with the wave at velocity c_-.

3. Properties of conjugate flows

We consider the possibility that the primary flow as defined in § 2, say flow A, is connected through some form of steady transition to another axially uniform flow B with the same cross-section σ. Mass and available energy are assumed to be conserved through the transition, so that flow B may be termed conjugate to A and vice versa. The nature of the transition will be discussed later, after we have worked out the relationships between the pair of conjugate flows.

Since flow B is like A axially uniform, the pressure in it is constant. Hence, on the understanding that the co-ordinates x, y of a streamline in flow A become x', y' in B, the density of the fluid in flow B is given by the function R introduced in (2.1), thus

$$\rho(x', y') = R(p, x, y).$$

Let $w(x', y')$ denote the axial velocity in flow B. Then the condition of mass conservation may be written

$$R(0)\, W = R(p)\, w\, J, \tag{3.1}$$

where

$$J = \partial(x', y')/\partial(x, y)$$

is the ratio $dx'\, dy'/dx\, dy$ between the cross-sectional areas of the same elementary stream-tube in flow B and in flow A. We assume that $w > 0$ everywhere in σ (i.e. there is no stagnation in flow B), and that the values of $R(p)$ in σ are bounded positively from below (i.e. we exclude the possibility of cavitation, which might occur for sufficiently large negative values of p). Thus J is determined by (3.1) as a bounded function.

The condition of energy conservation is, from (2.4),

$$\tfrac{1}{2}W^2 = \tfrac{1}{2}w^2 + H(p), \tag{3.2}$$

where $H(p)$ is the enthalpy function introduced in (2.2). Using this to eliminate w from (3.1), we obtain

$$\left.\begin{aligned} J &= \frac{R(0)}{R(p)} \frac{W}{\{W^2 - 2H(p)\}^{\frac{1}{2}}} \\ &= I(p), \quad \text{say.} \end{aligned}\right\} \tag{3.3}$$

[The function $I(p)$, like $R(p)$ and $H(p)$, is implied to depend also on x, y.] Since

$$\int_\sigma J\,dx\,dy = \int_\sigma dx'\,dy' = \mathscr{A},$$

the condition determining the possible value of p in flow B is hence seen to be

$$\left.\begin{aligned} \mathscr{A} &= \int_\sigma \frac{R(0)}{R(p)} \frac{W}{\{W^2 - 2H(p)\}^{\frac{1}{2}}}\,dx\,dy \\ &= \mathscr{I}(p), \quad \text{say.} \end{aligned}\right\} \tag{3.4}$$

Anticipating that equation (3.4) has only one non-trivial root, we denote it by p_1.

To find flow B in detail, we would need to solve the differential equation $J = I(p_1)$ in σ, subject to the obvious boundary condition that the vector $(x', y') - (x, y)$ has no normal component at the boundary of σ. The most important properties of flow B in relation to flow A can, however, be deduced directly from equation (3.4).

The condition for sonic flow

As recalled at the end of § 2, a flow is called sonic when an infinitesimal wave of extreme length can be superposed upon it. Accordingly, this condition may be inferred to apply to flow A in the limit as the root p_1 of (3.4) tends to zero – that is, as the conjugate flows A and B coincide. Thus the sonic condition for the primary flow is expressible by

$$\mathscr{I}'(0) = 0,$$

which, when we use the fact that $H'(p) = 1/R(p)$ by definition, is found to be the same as

$$\int_\sigma \frac{dx\,dy}{R(0)\,W^2} = \int_\sigma \frac{R'(0)}{R(0)}\,dx\,dy \tag{3.5}$$

[cf. (3.8) below]. It may naturally be assumed that the density of the fluid increases with pressure, so that $R'(0)$ as well as $R(0)$ is positive everywhere in σ.

Putting $W = c$ (const.) in (3.5), we get a formula for the speed c of very long infinitesimal waves (i.e. the low-frequency sound speed) relative to the fluid at rest, thus

$$c^2 = \left(\int_\sigma \frac{dx\,dy}{R(0)}\right) \div \left(\int_\sigma \frac{R'(0)}{R(0)}\,dx\,dy\right). \tag{3.6}$$

It may be helpful to rewrite this formula with the meaning of the function R made explicit. Using zero subscripts to connote the undisturbed state of the fluid, we have

$$c^2 = \left(\int_\sigma \frac{1}{\rho_0}\,dx\,dy\right) \div \left(\int_\sigma \frac{1}{\rho_0}\left(\frac{d\rho}{dp}\right)_0 dx\,dy\right). \tag{3.6'}$$

In the case that the properties of the fluid are uniform over the cross-section, this reduces to the familiar result $c^2 = (dp/d\rho)_0$.

From the considerations in § 2 it is evident that for a general primary velocity $W(x,y)$, the wave velocities c_+ and c_- entailed in the definition (2.5) are roots of

$$\int_\sigma \frac{dx\,dy}{R(0)\,(W-c_\pm)^2} = \int_\sigma \frac{R'(0)}{R(0)}\,dx\,dy. \tag{3.7}$$

If the positive function W has bounded derivatives in a neighbourhood of its maximum on σ, the root c_+ satisfies

$$c_+ > \max_{x,y\in\sigma} W(x,y) > 0,$$

since the value of the integral on the left-hand side of (3.7) can be varied over the interval $(0,\infty)$ by choice of c_+ satisfying this condition. The primary flow is subsonic ($N_A < 1$) if $c_- < 0$, which is implied if

$$\int_\sigma \frac{dx\,dy}{R(0)\,W^2} > \int_\sigma \frac{R'(0)}{R(0)}\,dx\,dy.$$

Hence, in view of the fact that

$$\mathscr{I}'(0) = \int_\sigma \left\{\frac{1}{R(0)\,W^2} - \frac{R'(0)}{R(0)}\right\} dx\,dy, \tag{3.8}$$

it follows that

$$\mathscr{I}'(0) > 0 \quad \text{implies } \textit{subsonic} \quad (N_A < 1). \tag{3.9}$$

Similarly it appears that

$$\mathscr{I}'(0) < 0 \quad \text{implies } \textit{supersonic} \quad (N_A > 1). \tag{3.10}$$

The transcritical property

We proceed on the assumption that

$$\mathscr{I}''(p) > 0 \tag{3.11}$$

for all relevant values of p. Useful conclusions may still be reached without this assumption, but its introduction is justified since it simplifies the theory and is in fact borne out in the application to suspensions of gas bubbles. Considering this application in § 4, we shall find that $I''(p) > 0$, where $I(p)$ is the integrand of the integral $\mathscr{I}(p)$ over σ, so that the condition (3.11) is obviously provided.

Accordingly, if $\mathscr{I}'(0) < 0$ and thus the primary flow A is supersonic, the following implications of equation (3.4) become clear as illustrated in figure 1. The non-trivial root p_1 is evidently unique and it must be positive (i.e. pressure is increased in the conjugate flow B). Furthermore

$$\mathscr{I}'(p_1) > 0. \tag{3.12}$$

We now show that this inequality implies flow B to be subsonic ($N_B < 1$).

Upon differentiation of (3.4) and use of the fact that $H'(p) = 1/R(p)$, it appears that

$$\mathscr{I}'(p_1) = \int_\sigma \left\{\frac{1}{R(p_1)\,w^2} - \frac{R'(p_1)}{R(p_1)}\right\} \frac{R(0)\,W}{R(p_1)\,w}\,dx\,dy$$

and hence

$$\mathscr{I}'(p_1) = \int_\sigma \left\{\frac{1}{R(p_1)\,w^2} - \frac{R'(p_1)}{R(p_1)}\right\} dx'\,dy' \tag{3.13}$$

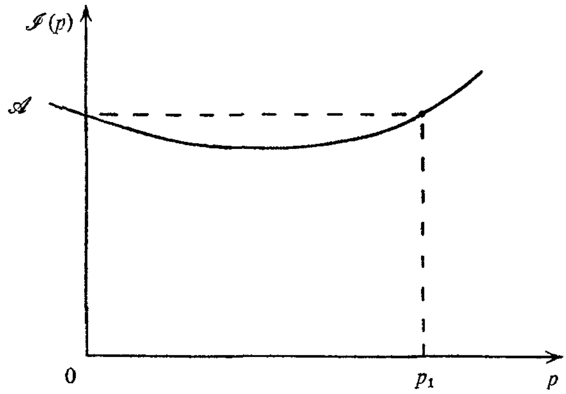

FIGURE 1. Illustration of equation (3.4).

by virtue of the mass-conservation condition (3.1). But we know that

$$R(p_1) = \rho(x', y'),$$

and similarly
$$R'(p_1) = \left[\frac{d\rho(p, x', y')}{dp}\right]_{p=p_1}$$

With regard to flow B, therefore, (3.12) is seen in the light of (3.13) to be precisely the criterion of subsonic flow that we exemplified by (3.8) and (3.9) with regard to flow A.

Complementary conclusions for the case when the primary flow is subsonic can be obtained by an obvious adaption of the argument, or simply by rephrasing the previous conclusions after redefining B as the primary flow. If $\mathscr{I}'(0) > 0$, then $p_1 < 0$ and $\mathscr{I}'(p_1) < 0$, which implies that the conjugate flow B is supersonic.

As thus exemplified, the transcritical property of conjugate flows has appeared to be a universal attribute of such flows in analogous problems (see Benjamin 1971, p. 589).

Flow force

The flow force is defined as the sum axial pressure force and momentum flux through the whole cross-section of the flow, thus

$$S = p\mathscr{A} + \int_\sigma \rho w^2 \, dx' \, dy'. \tag{3.14}$$

Using the condition of mass conservation (3.1), we may express this alternatively by

$$S = p\mathscr{A} + \int_\sigma R(0) \, Ww \, dx \, dy,$$

and hence, using the condition of energy conservation, by

$$S = p\mathscr{A} + \int_\sigma R(0) \, W\{W^2 - 2H(p)\}^{\frac{1}{2}} \, dx \, dy. \tag{3.15}$$

The last expression depends only on p and the primary-flow specifications. Differentiating it with respect to p, we obtain

$$dS/dp = \mathscr{A} - \mathscr{I}(p), \tag{3.16}$$

where $\mathscr{I}(p)$ is the integral in equation (3.4). Thus we see that $dS/dp = 0$ for either of the conjugate flows A or B, i.e. for $p = 0$ or $p = p_1$. This result exemplifies a general principle that has been established in studies of various other conjugate-flow problems: namely, if an expression for flow force is varied subject to conditions of mass and energy conservation, it takes stationary values for conjugate flows (cf. Benjamin 1971, §§ 3.5, 4, 6.5).

As a reference to figure 1 makes clear, the result (3.6) implies that when the primary flow is supersonic and consequently p_1 is positive, then $dS/dp > 0$ for $0 < p < p_1$. Correspondingly, when the primary flow is subsonic and p_1 is negative, then $dS/dp > 0$ for $p_1 < p < 0$. It follows that in any pair of conjugate flows, the subsonic member has the greater value of flow force. This proposition too has been found to apply generally in various other conjugate-flow problems.

Interpretation

In practice a transition between conjugate flows of the present kind may be exemplified in either of two ways, which we can appreciate by analogy with other conjugate-flow systems. The case admitting simpler interpretation occurs when the primary flow is subsonic. A transition to the conjugate supersonic flow may then be brought about by an obstacle placed in the stream, whose action is thus analogous to that of a sluice-gate spanning an open-channel flow. The crux of this situation is that the relative deficiency of flow force in the supersonic flow downstream corresponds to the external force required to hold the obstacle in place.

A transition from a supersonic primary flow to its subsonic conjugate serves as a rudimentary model for a normal shock wave. It cannot occur without modification, however, because of the unbalance of flow force. In a strong shock dissipative effects are likely to be predominant, so that the present theory would be irrelevant; but for a weak shock in a *dispersive* system of the present kind, such as a gas-bubble suspension (see § 5 below), dissipation may be comparatively unimportant and the flow-force excess of the subsonic flow may be balanced by oscillatory-wave formation on an intermediate length scale. Thus the hypothetical conjugate is significant in that the actual wavy flow subsists on it by consuming its extra flow force as 'wave resistance'. This idea serves to explain undular hydraulic jumps, for example, and it has been applied to the interpretation of the vortex breakdown phenomenon (Benjamin 1962, 1965, 1967). A theory that can be used to account more precisely for undular shocks in homogeneous suspensions of gas bubbles has been developed by van Wijngaarden (1968).

Dissipative transitions

In real applications, as to non-uniform suspensions of gas bubbles, it would be a complicated problem to calculate the various secondary effects not included in the idealized model considered so far. The thermodynamic processes in the fluid may not be entirely reversible, and, with regard to suspensions, dissipative effects may be expected to arise particularly from relative motion between the bubbles and the liquid. A precise account of such effects is beyond the scope of the present type of theory, which is concerned only with broad principles governing flow transitions and not with their internal structure. However, to show that frictional dissipation and other irreversible effects are broadly accountable within the framework of present ideas, the following hypothetical modification is instructive.

As introduced in § 2, the non-negative quantity Δ is the excess of internal energy above the value achieved through the reversible adiabatic processes previously represented, and for the conjugate flow in question we suppose that

$$\Delta = \epsilon f(x, y),$$

where $f(x, y)$ is a particular non-negative function on σ. That is, we take a hypothetical spatial distribution for the waste of available energy, but allow its magnitude ϵ to be an adjustable parameter. (A guess such as $f = W^2$ might be made to specify the model further, but an *a priori* estimate of f is essentially beyond present means.) In place of (3.2), the condition of energy conservation now becomes

$$\tfrac{1}{2}W^2 = \tfrac{1}{2}w^2 + H(p) + \epsilon f,$$

and we assume that ϵ is small enough to keep $w > 0$ as before. Hence the expression (3.15) for flow force is replaced by

$$S(p, \epsilon) = p\mathscr{A} + \int_\sigma R(0)\, W\{W^2 - 2H(p) - 2\epsilon f\}^{\frac{1}{2}}\, dx\, dy, \tag{3.17}$$

and we note that in the relevant range of the two variables p and ϵ (i.e. such that $w > 0$), the partial derivative S_ϵ is negative while $S_{p\epsilon}$ is positive. Since

$$\left(\frac{d\epsilon}{dp}\right)_{S=\text{const.}} = -\frac{S_p}{S_\epsilon},$$

this implies that the variation of ϵ with p, subject to S keeping its original value $S(0, 0)$, is in the same sense as the variation of S with p when $\epsilon = 0$. Thus we may expect that the flow-force excess found to characterize a subsonic conjugate flow when available energy is conserved can be cancelled by dissipation.

From (3.17) it appears that

$$S_p(p, \epsilon) = \mathscr{A} - \mathscr{I}(p, \epsilon), \tag{3.18}$$

where $\mathscr{I}(p, \epsilon)$ is the same as the function $\mathscr{I}(p)$ in (3.4) except that the radical representing w is replaced as in (3.17). As noted before, $S_p(0, 0)$ is identically zero. And, corresponding to (3.4) and (3.16), the condition of overall mass conservation is now

$$S_p(p, \epsilon) = 0. \tag{3.19}$$

Assuming the primary flow to be supersonic, i.e.

$$S_{pp}(0, 0) = -\mathscr{I}_p(0, 0) > 0, \tag{3.20}$$

we consider the possibility of a conjugate flow conserving flow force, being therefore realizable through a stationary shock. That is, values of p and ϵ are required satisfying both (3.19) and

$$S(p, \epsilon) = S(0, 0). \tag{3.21}$$

If we suppose as before [cf. (3.11)] that

$$S_{ppp}(p,\epsilon) = -\mathscr{I}_{pp}(p,\epsilon) < 0,$$

and take account of (3.20) together with the fact that S_ϵ is negative, a diagram of the smooth function $S(p,\epsilon)$ shows that a solution (p_1,ϵ_1) always exists with $p_1 > 0$ and $\epsilon_1 > 0$. It appears, moreover, that

$$S_{pp}(p_1,\epsilon_1) = -\mathscr{I}_p(p_1,\epsilon_1) < 0; \tag{3.22}$$

and on the basis of this fact the argument used before, in relation to (3.12) and (3.13), confirms that the conjugate flow is again subsonic. Thus we find all the usual properties of a normal shock: a transition from a supersonic to a subsonic state of flow conserving flow force, with a rise of pressure and a loss of available energy.

The relationship between the available-energy conserving and the flow-force conserving conjugate flows respective to the same supersonic primary flow is particularly easy to appreciate in the case of a weak shock $(0 < N_A - 1 \ll 1)$, for which

$$S_{pp}(0,0) = \delta, \quad \text{say},$$

is a small positive number, very much less than the positive numbers

$$-S_{ppp}(0,0) = \alpha, \quad -S_\epsilon(0,0) = \beta.$$

The leading terms of the Taylor-series expansion of S then give approximately

$$S(p,\epsilon) = S(0,0) + \tfrac{1}{2}\delta p^2 - \tfrac{1}{6}\alpha p^3 - \beta\epsilon.$$

In the first instance, we take $\epsilon = 0$ and the condition $S_p = 0$ at $p = p_1$ determines

$$p_1 = 2\delta/\alpha > 0,$$

and hence

$$S(p_1) - S(0) = \tfrac{2}{3}\delta^3/\alpha^2 = \tfrac{1}{12}\alpha p_1^3.$$

Furthermore we have

$$S_{pp}(p_1) = -\delta,$$

which fixes the subsonic Mach number N_B of the conjugate flow. In the second instance, when both the conditions (3.19) and (3.20) are applied, the pressure p_1 and N_B are the same as before, but now

$$\epsilon_1 = \alpha p_1^3/12\beta.$$

Finally, we note that, when the primary flow is subsonic, dissipation augments the flow-force deficiency in the supersonic conjugate flow. Thus, as may be expected, the drag experienced by an obstacle bringing about a subsonic–supersonic transition is increased by dissipation. Alternatively, a given external force can cause such a transition only if the accompanying dissipation is not too large.

4. Application to suspensions of gas bubbles in liquids

To substantiate this application of the general theory, it is required to specify the functions $R(p)$ and $H(p)$ introduced in (2.1) and (2.2). Descriptions of the essential properties of gas-bubble suspensions are available in many places, for example, in Prandtl (1952, pp. 330, 331), Mallock (1910), Carstensen & Foldy (1947), Meyer & Skudzrijk (1953) or Hsieh & Plesset (1961), and we shall proceed on the usual assumption that the bubbles are small enough and numerous enough for local aggregate properties to be meaningful. It is also usually assumed that the bubbles are locally all equal, but allowance will be made here for the more realistic case where there is a distribution of bubble sizes. Although the main concern of the theory is that the density ρ and other properties of the composite fluid vary with position in the cross-section σ, our immediate task is to evaluate them locally, as if for a homogeneous mixture.

In the application of conjugate-flow concepts, the possibility of relative motion between the bubbles and the liquid is disregarded, justifiably in respect of the separate states of flow with uniform pressures. It should be acknowledged, however, that in passing through a transition of the kind approximately modelled by the theory, a gas-bubble suspension will inevitably suffer axial pressure gradients that will induce some such relative motion; and, as was suggested earlier, this may be a major cause of viscous dissipation tending to invalidate the postulated condition of energy conservation between conjugate states. The expansive or contractive flow created in the neighbourhood of each bubble by dilatations of the suspension is probably less significant as a cause of dissipation, although its inertia is responsible for frequency-dispersive effects which may be manifest in undular shocks (further reference to these effects will be made in the penultimate paragraph of § 5). These various factors are ignored here in the same spirit as previous applications of conjugate-flow principles: the point is that in reducing the problem to the present manageable essentials there remains a theoretical framework on which plausible interpretations of observed phenomena may eventually be built.

According to estimates that have been made in previous theoretical studies of gas-bubble suspensions (e.g. see Ackeret 1930; Campbell & Pitcher 1958; Hsieh & Plesset 1961), there is reason to assume that changes in the gas accompanying expansions or compressions of the suspension are *isothermal*, provided the bubbles are fairly small (say, with diameters 0·1 mm or less) and the hydrodynamic time scales are not very short (say, longer than 1 millisecond). The implication is that the conductivities of liquid and gas are large enough, while the flow processes are slow enough, for thermal equilibrium to be virtually achieved at each instant along a fluid trajectory. We shall proceed on this assumption, noting that the expression (2.2) for reversible enthalpy changes of the *composite* fluid is consistent with it. It should be acknowledged that, in some practical examples with larger bubbles, heat exchange between the gas and the surrounding liquid may lag significantly behind the flow-induced changes in the suspension, even perhaps to the extent that the assumption of adiabatic conditions in the gas might provide a better model; and for conditions intermediate between this last extreme and the case assumed, the expression (2.2) ceases to identify enthalpy.

Let v denote the volume of gas suspended per unit mass of liquid. Ignoring the mass of the gas, we take this to be the same as the volume of gas per unit mass of the mixture. Then, if ρ_l denotes the density of the liquid and s the volume fraction of gas in the mixture, which is the same as ρv, the density of the mixture is evidently given by

$$\rho = \rho_l(1-s) = \rho_l(1-\rho v), \tag{4.1}$$

from which follows

$$\rho = \frac{\rho_l}{1+\rho_l v}. \tag{4.2}$$

When the gas is contained in spherical bubbles of equal radius a, we have simply

$$v = \tfrac{4}{3}\pi n a^3,$$

where n is the number of bubbles per unit mass. More generally, for a suspension of various spherical bubbles, we have

$$v = \tfrac{4}{3}\pi \sum_{i=1}^{n} a_i^3. \tag{4.3}$$

While being perfectly clear in principle, this representation of v would not be particularly useful for practical evaluation when – as is implied by our basic assumption about the suspension – the bubbles are very profuse. A more expedient representation in terms of probabilities or averages over fairly large samples might be

$$v = \tfrac{4}{3}\pi \int_0^\infty a^3\, d\xi(a), \tag{4.4}$$

where $\xi(a)$ is the probable (or average) number of bubbles with radii not greater than a, per unit mass of the mixture. We shall, however, use the simpler representation (4.3) in what follows, conveniently leaving the limits $1, n$ implicit: thus $v = \frac{4}{3}\pi\Sigma a_i^3$, where the summation Σ carries the dimension $(\text{mass})^{-1}$.

Let P denote *absolute* pressure and P_0 its value in the primary state of the suspension, so that in accord with the definition of p in § 2 we have

$$P = P_0 + p.$$

We assume that $P > 0$ always, excluding the possibility of unlimited expansion of any bubble (i.e. cavitation). For any particular bubble the pressure Q_i of the contained gas is given by

$$Q_i = P + (2\gamma/a_i), \tag{4.5}$$

where γ is the coefficient of surface tension for the spherical gas–liquid interface.† By the assumption of isothermal conditions in the gas, the product of each Q_i and the respective bubble volume is constant. Thus (4.5) shows that

$$\begin{aligned}(P_0+p)\,a_i^3 + 2\gamma a_i^2 &= \text{const.}\\ &= P_0\, a_{i0}^3 + 2\gamma a_{i0}^2. \end{aligned}\tag{4.6}$$

† The pressure difference $Q_i - P$ due to surface tension appears unlikely to be important in many practical examples of bubbly liquids, even when surface tension has an important rôle in keeping the bubbles spherical. For instance, if P is 1 bar and the bubble radius a_i is 0·1 mm, which in most practical situations would not be an exceptionally large value, then the fractional difference $(Q_i - P)/P$ is only 0·014 for water. The effect of surface tension is worth including in the theory, however, because it poses little extra difficulty and because applications are possible where the bubbles are small enough or the pressure of the suspension is low enough for this effect to become significant.

If $P_0+p > 0$, this cubic for a_i has a single positive root: that is, a_i is uniquely determined as a function of p. This function can be expressed explicitly but does not warrant that much trouble, and we may be content to connote it by writing

$$a_i = a_i(p) \quad [a_i(0) \equiv a_{i0}].$$

By substitution of this result in (4.3) and (4.2), the function of p determining the local density ρ is seen to be

$$R(p) = \frac{\rho_l}{1+\frac{4}{3}\pi\rho_l \Sigma a_i^3(p)}, \tag{4.7}$$

on the assumption that ρ_l is independent of p (i.e. the compressibility of the liquid is neglected in comparison with that of the mixture).

In the case that the bubbles are locally all equal, (4.7) obviously reduces to

$$R(p) = \frac{\rho_l}{1+\frac{4}{3}\pi n\rho_l a^3(p)}. \tag{4.8}$$

If the effect of surface tension is ignored, the variety of the bubble sizes becomes immaterial, for (4.6) and (4.3) imply that $v = P_0 v_0/(P_0+p)$ – as is directly evident since Pv is the total gas constant per unit mass of the mixture and is therefore invariant. The density function is then

$$R(p) = \frac{\rho_l(P_0+p)}{P_0(1+\rho_l v_0)+p}. \tag{4.9}$$

Note that each of these three expressions for $R(p)$ has the foreseeable property $R \to \rho_l$ as $p \to \infty$.

For a *homogeneous* suspension at rest in its primary state (i.e. $W \equiv 0$), the low-frequency sound speed c is given by $c^{-2} = R'(0)$. From (4.7) we have

$$\frac{R'(0)}{\rho_l} = -\frac{4\pi\rho_l \Sigma a_{i0}^2 a_i'(0)}{(1+\rho_l v_0)^2},$$

and differentiation of (4.6) shows that

$$a_i'(0) = -\frac{a_{i0}^2}{3P_0 a_{i0}+4\gamma}.$$

Hence, using the identity $\rho_l v = s(1-s)$ implied by (4.1), we obtain

$$\frac{R'(0)}{\rho_l} = s_0(1-s_0)\left\{\frac{\Sigma a_{i0}^3[P_0+(4\gamma/3a_{i0})]^{-1}}{\Sigma a_{i0}^3}\right\}. \tag{4.10}$$

In the case that all bubbles have the same radius a_0 in the primary state, this gives

$$c^2 = \frac{1}{R'(0)} = \frac{P_0+(4\gamma/3a_0)}{\rho_l s_0(1-s_0)}, \tag{4.11}$$

a formula that long had been known (see, for example, Mallock 1910). If the effect of surface tension is ignored, the result irrespective of variety of bubble sizes is

$$c^2 = \frac{P_0}{\rho_l s_0(1-s_0)}. \tag{4.12}$$

Consider now the general formula (3.6) for the sound speed when the properties of the composite fluid are not uniform over the cross-section σ. Taking the case where the bubbles are locally of the same size, so that $R'(0)$ is given by (4.11), we have

$$c^2 = \frac{P_0}{\rho_l}\left(\int_\sigma \frac{dx\,dy}{1-s_0}\right) \div \left(\int_\sigma \frac{s_0\,dx\,dy}{1+(\beta/a_0)}\right), \tag{4.13}$$

where $\beta = 2\gamma/P_0$. If $\beta = 0$, this expression is also valid independently of the local distribution of bubble sizes. In a practical case it is quite likely that $s_0 \ll 1$ everywhere in σ, and thus the density ρ is not significantly different from the density ρ_l of the mixture. If in addition surface tension is insignificant, (4.13) reduces approximately to

$$c^2 = \frac{P_0}{\rho_l \bar{s}_0},$$

where

$$\bar{s}_0 = \frac{1}{\mathscr{A}}\int_\sigma s_0\,dx\,dy$$

is the average volume fraction of gas in the duct; and so c is independent of how the total gas content of the mixture is distributed. Thus it appears that the low-frequency sound speed relative to fluid at rest will not be appreciably affected by non-uniformities in gas concentration unless surface tension is important or the volume fraction of gas is large enough to make the density of the mixture significantly less than that of the liquid.

The enthalpy function $H(p)$ can be found from (4.7), thus

$$H(p) = \int_0^p \frac{dp'}{R(p')} = \frac{p}{\rho_l} + \tfrac{4}{3}\pi\Sigma\int_0^p a_i^3(p')\,dp'.$$

Rearranging (4.6) and writing B_i for the constant on its right-hand side, we have

$$p = \frac{B_i}{a_i^3} - \frac{2\gamma}{a_i^2} - P_0,$$

from which there follows

$$\begin{aligned}\int_0^p a_i^3\,dp' &= \int_{a_{i0}}^{a_i}\left(2\gamma a - \frac{3B_i}{a}\right)da \\ &= \gamma(a_i^2 - a_{i0}^2) + B_i \ln(a_{i0}^3/a_i^3) \\ &= \gamma(a_i^2 - a_{i0}^2) + B_i \ln\left\{\frac{P_0 + p + (2\gamma/a_i)}{P_0 + (2\gamma/a_{i0})}\right\}.\end{aligned}$$

Hence the final result is

$$H(p) = \frac{p}{\rho_l} + \tfrac{4}{3}\pi\Sigma\left[\gamma\{a_i^2(p) - a_{i0}^2\} + (P_0 a_{i0}^3 + 2\gamma a_{i0}^2)\ln\left\{\frac{P_0 + p + [2\gamma/a_i(p)]}{P_0 + (2\gamma/a_{i0})}\right\}\right]. \tag{4.14}$$

In the case of equal bubbles, (4.14) reduces to

$$H(p) = \frac{p}{\rho_l} + \tfrac{4}{3}\pi n\left[\gamma\{a^2(p) - a_0^2\} + (P_0 a_0^3 + 2\gamma a_0^2)\ln\left\{\frac{P_0 + p + [2\gamma/a(p)]}{P_0 + (2\gamma/a_0)}\right\}\right]; \tag{4.15}$$

and if the effect of surface tension is ignored the result is

$$H(p) = \frac{p}{\rho_l} + P_0 v_0 \ln\left(\frac{P_0+p}{P_0}\right) \tag{4.16}$$

(cf. Prandtl 1952, p. 331).

The formulae (4.7) and (4.14), or the simplified versions noted below them, provide for the practical use of the theory given in § 3. For thereby the required properties of the composite fluid are calculable from data about the primary distribution of bubbles. Note again, as in the discussion of (4.3) and (4.4), that it may be convenient to replace the summations in (4.7) and (4.14) by integrals with respect to a probability measure $\xi(a)$.

Convexity of the function $I(p)$

To confirm the applicability of the conclusions established in § 3, in particular concerning the transcritical property of conjugate flows, it remains to show that $I''(p) > 0$, where $I(p)$ is the integrand in (3.4). In terms of the dimensionless variable

$$r = \rho_l/R = (1-s)^{-1},$$

this function may be expressed as

$$I(p) = \frac{WR(0)\,r(p)}{\rho_l\sqrt{\{W^2-2H(p)\}}},$$

and we recall that the radical in the denominator represents w, the axial velocity in the conjugate flow. Differentiating this expression with respect to p and using the fact that $H'(p) = 1/R(p) = r(p)/\rho_l$ by definition, we find that

$$I'' = \frac{WR(0)}{\rho_l w}\left\{r'' + \frac{3rr'}{\rho_l w^2} + \frac{3r^3}{(\rho_l w^2)^2}\right\}. \tag{4.17}$$

It is assumed that $W > 0$ everywhere and, as explained in § 3, we restrict attention to a range of pressure such that $w > 0$, thus presuming no stagnation to occur in the conjugate flow. The required property $I'' > 0$ is therefore proved by showing that

$$f = r'' + \frac{3rr'}{\rho_l w^2} + \frac{3r^3}{(\rho_l w^2)^2} > 0.$$

Now r is positive and r' negative always (i.e. the density of the suspension always increases upon an increase of pressure). It follows that for arbitrary variations of $\rho_l w^2$ the preceding expression for f has a minimum value, which is given when $\rho_l w^2 = -2r^2/r'$. Evidently f cannot be less than this value, whatever the value of $\rho_l w^2$ may in fact be, and thus we deduce that $I'' > 0$ is implied by

$$rr'' - \tfrac{3}{4}r'^2 > 0. \tag{4.18}$$

To test this inequality in the present example, we have from (4.7)

$$r = 1 + \kappa\Sigma a_i^3 \quad \text{with} \quad \kappa = \tfrac{4}{3}\pi\rho_l,$$

and hence

$$r' = 3\kappa\Sigma a_i^2 a_i',$$

$$r'' = 3\kappa\Sigma(a_i^2 a_i')'.$$

Differentiation of (4.6) with respect to p gives at once

$$a_i^2 a_i' = -\frac{a_i^3}{3P+4(\gamma/a_i)},$$

and then after a little calculation

$$(a_i^2 a_i')' = \frac{\{18P+28(\gamma/a_i)\}\, a_i^3}{\{3P+4(\gamma/a_i)\}^3}.$$

Thus we obtain

$$rr'' - \tfrac{3}{4}r'^2 = r'' + 6\kappa^2(\Sigma a_i^3)\left(\Sigma \frac{\{9P+14(\gamma/a_i)\}\, a_i^3}{\{3P+4(\gamma/a_i)\}^3}\right) - \tfrac{27}{4}\kappa^2\left(\Sigma \frac{a_i^3}{3P+4(\gamma/a_i)}\right)^2, \tag{4.19}$$

in which $r'' > 0$. Use is now made of Cauchy's inequality for series (Hardy, Littlewood & Polya 1952, p. 16). Since $a_i > 0$ for all $i = 1, 2, \ldots, n$, we have

$$\left(\Sigma \frac{a_i^3}{3P+4(\gamma/a_i)}\right)^2 \leqslant (\Sigma a_i^3)\left(\Sigma \frac{a_i^3}{\{3P+4(\gamma/a_i)\}^2}\right),$$

and hence conclude from (4.19) that

$$rr'' - \tfrac{3}{4}r'^2 \geqslant r'' + \tfrac{3}{4}\kappa^2(\Sigma a_i^3)\, \Sigma \frac{\{45P+76(\gamma/a_i)\}\, a_i^3}{\{3P+4(\gamma/a_i)\}^3}.$$

This result obviously establishes the inequality (4.18), which verifies the positivity of I'' given by (4.17).

5. Conclusion

The main substance of this paper is the demonstration that conjugate-flow principles, as have appeared in studies of various other hydrodynamical problems, apply to the problem of heterogeneous fluid flow explained in §§2 and 3, so affording a rationale for a diversity of possible phenomena. As is already known with regard to open-channel flows, axisymmetric vortex flows and flows of density-stratified fluids under gravity – to name the most discussed examples – it has again been confirmed that any energy-conserving transition between axially uniform flows goes from a supercritical to a subcritical state or vice versa, and that in such a pair of conjugate flows the subcritical member always has the greater value of flow force. The abstract mathematical reasons that explain these universal properties of nonlinear flow systems have been pointed out by Benjamin (1971), and the present conclusions might otherwise be anticipated by broad physical analogies with other, more familiar long-wave problems. But the actual way in which the conjugate-flow aspects of this problem emerge seems novel.

The application to non-uniform suspensions of gas bubbles may offer some prospect of practical use. The shortage of relevant experimental results, however, even for approximately homogeneous suspensions, suggests that the special effects described by the theory may prove exceedingly difficult to appreciate in practice. For given non-uniformities of bubble concentration the *results of* the theory can be evaluated without much trouble, but the temptation to present a comprehensive calculation for some arbitrary model has been resisted. Such an exercise would seem hardly justified until a specific experimental situation is in view.

Interesting and challenging new aspects of the problem arise if the restriction to very long waves is abandoned. It can at once be recognized that two mechanisms of dispersion are operative. The first entails the inertia of the radial motion of liquid in the vicinity of each pulsating bubble, producing a phase lag between the local ambient pressure in the suspension and the gas pressure inside the bubbles. This effect was noted by Carstensen & Foldy (1947) and described in more detail by Meyer & Skudzrijk (1953). Recently van Wijngaarden (1968), considering waves of small but finite amplitude and moderate length in homogeneous suspensions, showed that the competition between this dispersive effect and first-order nonlinear effects is describable by equations similar to those for long water waves. The second dispersive mechanism depends on the structure of the inhomogeneities in fluid properties, and thus it poses a theoretical problem of a familiar kind concerning wave propagation in non-uniform media. A theory accounting for both kinds of dispersive effect and also nonlinear effects would probably be quite difficult.

We finally note a result that may be derived straightforwardly when the second type of dispersion is considered independently of the first. This relates to an infinitesimal wave, propagating in a heterogeneous fluid originally at rest, such that the pressure perturbation takes the form

$$p = \hat{p}(x,y)\, e^{ik(z-ct)}.$$

The wave speed c and wavenumber k are found to be related through the characteristic-value problem

$$\left.\begin{aligned} \nabla.(r\nabla\hat{p}) + k^2 r(1-c^2 q)\,\hat{p} = 0 \quad \text{in } \sigma, \\ \partial_n \hat{p} = 0 \quad \text{on boundary of } \sigma, \end{aligned}\right\} \qquad (5.1)$$

where $r = \rho_l/R(0,x,y)$, $q = R'(0,x,y)$ and ∂_n denotes the normal derivative. If $k^2 = 0$, the only solution of (5.1) is $p = \text{const}$. But the requirement that (5.1) should have a non-trivial solution for arbitrarily small k^2 [in fact taking the form $1 + k^2\hat{p}_1(x,y) + O(k^4)$] is readily seen to determine the long-wave limit $c(0)$ of $c(k)$. It appears in this way that

$$\int_\sigma \{r - c^2(0)\, rq\}\, dx\, dy = 0,$$

which recovers the formula (3.6) for the low-frequency sound speed.

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